Quantum Game of Life

D. Bleh, T. Calarco and S. Montangero

Institut für Quanteninformationsverarbeitung Albert-Einstein-Allee 11 D-89069 Ulm, Germany.

PACS 03.67.-a – Quantum Information
PACS 89.75.-k – Complex systems
PACS 02.50.Le – Decision theory

Abstract – We introduce a quantum version of the Game of Life and we use it to study the emergence of complexity in a quantum world. We show that the quantum evolution displays signatures of complex behaviour similar to the classical one, however a regime exists, where the quantum Game of Life creates more complexity, in terms of diversity, with respect to the corresponding classical reversible one.

The Game of Life (GoL) has been proposed by Conway in 1970 as a wonderful mathematical game which can describe the appearance of complexity and the evolution of "life" under some simple rules [1]. Since its introduction it has attracted a lot of attention, as despite its simplicity, it can reveal complex patterns with unpredictable evolution: From the very beginning a lot of structures have been identified, from simple blinking patterns to complex evolving figures which have been named "blinkers", "gliders" up to "spaceships" due to their appearance and/or dynamics [2]. The classical GoL has been the subject of many studies: It has been shown that cellular automata defined by the GoL have the power of a Universal Turing machine, that is, anything that can be computed algorithmically can be computed within Conway's GoL [3,4]. Statistical analysis and analytical descriptions of the GoL have been performed; many generalisations or modifications of the initial game have been introduced as, for example, a simplified one dimensional version of the GoL and a semi-quantum version [5–7]. Finally, to allow a statistical mechanics description of the GoL, stochastic components have been added [8].

In this letter, we bridge the field of complex systems with quantum mechanics introducing a purely quantum GoL and we investigate its dynamical properties. We show that it displays interesting features in common with its classical counterpart, in particular regarding the variety of supported dynamics and different behaviour. The system converges to a quasi-stationary configuration in terms of macroscopic variables, and these stable configurations depend on the initial state, e.g. the initial density of "alive" sites for random initial configurations. We show that simple, local rules support complex behaviour and that the diversity of the structures formed in the steady state resembles that of the classical GoL, however a regime exists where quantum dynamics allows more diversity to be created than possibly reached by the classical one.

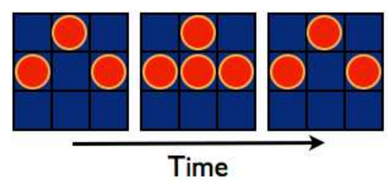

Fig. 1: Example of the evolution of the GoL described by Hamiltonian (1) for a simple initial configuration. Empty (blue) squares are "dead" sites, coloured (red) ones are "alive".

The universe of the original GoL is an infinite two-dimensional orthogonal grid of square cells with coordination number eight, each of them in one of two possible states, alive or dead [1]. At each step in time, the pattern present on the grid evolves instantaneously following simple rules: any dead cell with exactly three live neighbours comes to life; any live cell with less than two or more than three live neighbours dies as if by loneliness or overcrowding. As already pointed out in [7], the rules of the GoL are irreversible, thus their generalisation to the quantum case implies rephrasing them to make them compatible with a quantum reversible evolution. The system under study is a collection of two-level quantum systems, with two possible orthogonal states, namely the state "dead" ($|0\rangle$) and "alive" ($|1\rangle$). Clearly, differently from the classical case, a site can be also in a superposition of the two possible clas-

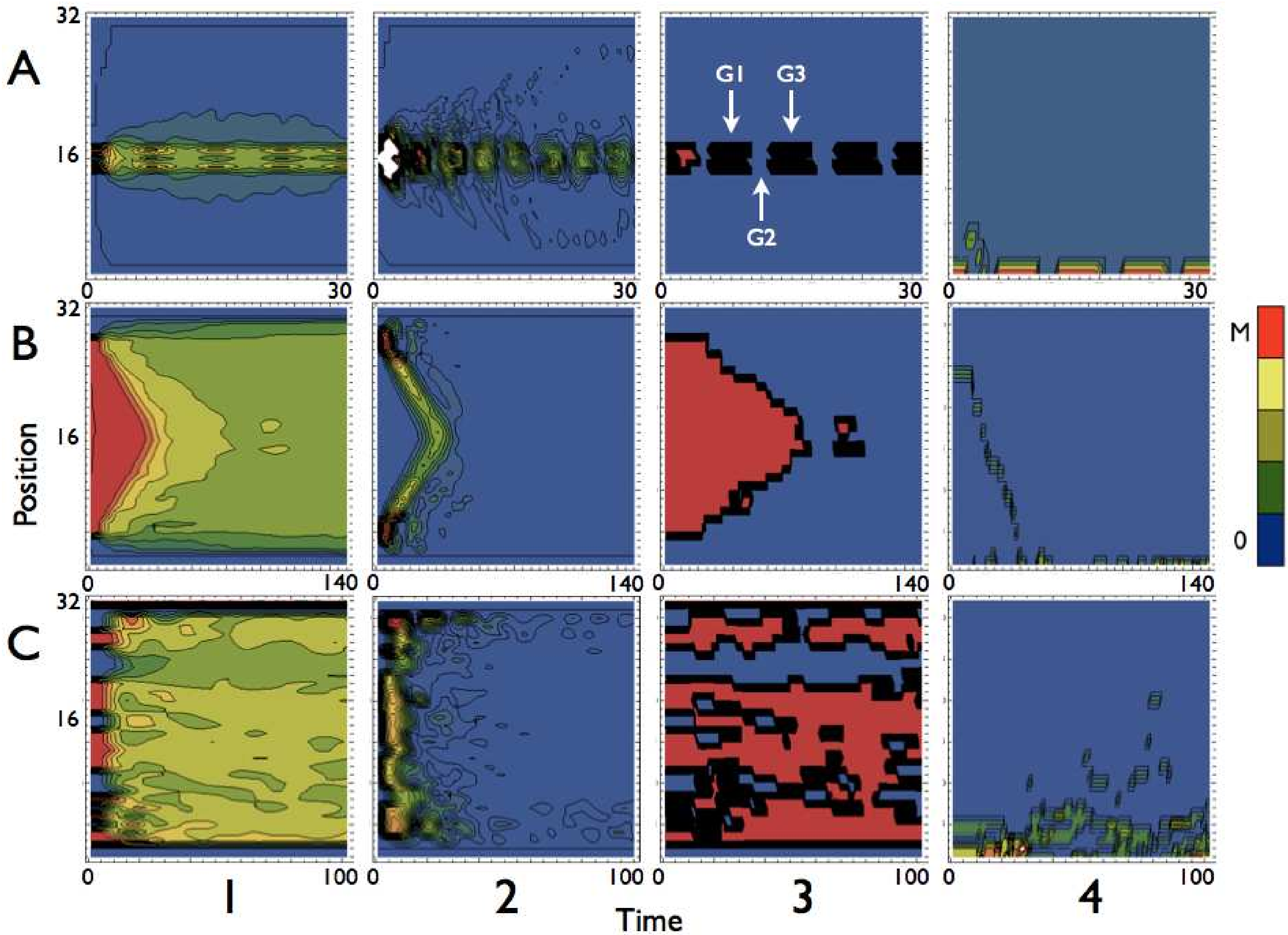

Fig. 2: Colour on-line. From left to right: Countour plot of the time evolution of the populations $\langle n_i(t)\rangle$ (column 1), visibility $v_i(t)$ (column 2), discretized populations $\mathcal{D}_i(t)$ (column 3) and clustering $\mathcal{C}(\ell,t)$ (column 4) for three different initial configurations: four alive sites separated by two dead ones (A), twenty-four alive sites grouped together (B) and a random initial configuration (C). Time is reported on the x-axis (in arbitrary units), and position (cluster size) $i = 1, \ldots, L$ on the y-axis in columns one to three (four). Arrows in panel 3A highlight the three subsequent generations of a "blinker" reported schematically in Fig. 3. The colour code goes from zero to $M = 1$ ($M = 4$ for the clustering and to $M = .1$ for the visibility), from blue through green to red.

sical states. The dynamics is defined as follows in terms of the GoL language: a site with two or three neighbouring alive sites is active, where active means that it will come to life and eventually die on a typical timescale T (setting the problem timescale, or time between subsequent generations). That is, if maintained active by the surrounding conditions, the site will complete a full rotation, if not, it is "frozen" in its state. Stretching the analogy with Conway's GoL to the limit, we are describing the evolution of a Virus culture: each individual undergoes its life cycle if the environment allows it, otherwise it hibernates in its current state and waits for conditions to change such that the site may become active again. This slight modification allows us to recover the reversibility of the dynamics and to introduce a quantum model that, as we shall see, reproduces most of the interesting complex behaviour of the classical GoL from the point of view of a classical observer. However, its evolution is purely quantum and thus we are introducing a tool that will allow to study the emergence of complexity from the quantum world.

Model. – The Hamiltonian describing the aforementioned model is given by

$$H = \sum_{i=3}^{L-2} (b_i + b_i^\dagger) \cdot \left(\mathcal{N}_i^3 + \mathcal{N}_i^2\right) \qquad (1)$$

where L is the number of sites; b and $b^\dagger$ are the usual annihilation and creation operators ($\hbar = 1$); the operators $\mathcal{N}_i^2 = \sum_P n_\alpha n_\beta \bar{n}_\gamma \bar{n}_\delta$ and $\mathcal{N}_i^3 = \sum_{P'} n_\alpha n_\beta n_\gamma \bar{n}_\delta$ ($n = b^\dagger b$, $\bar{n} = 1-n$, the indices $\alpha, \beta, \gamma, \delta$ label the four neighbouring sites) count the population present in the four neighbouring sites (the sum runs on every possible permutation P

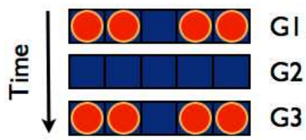

Fig. 3: Schematic representation of a one-dimensional time-evolution of the discretized population $\mathcal{D}_i(t)$ of a "blinker" (case A of Fig. 2). From left to right the states of subsequent generations are sketched. Empty (blue) squares are "dead" sites, coloured (red) ones are "alive"

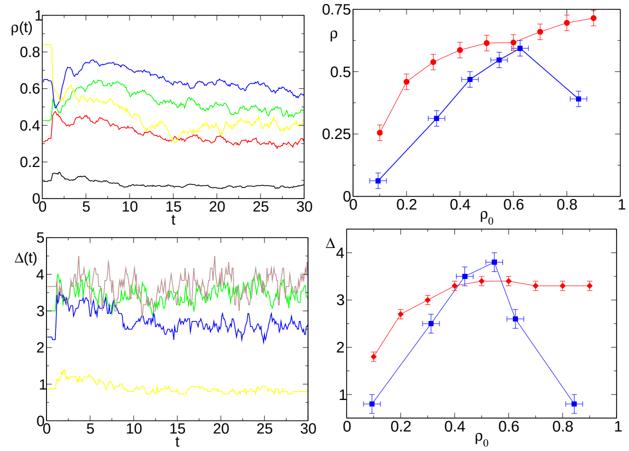

Fig. 4: Left: Average population $\rho(t)$ (upper) and diversity $\Delta(t)$ (lower) as a function of time for different initial population density ρ_0. Right: Equilibrium average population ρ (upper) and diversity Δ (lower) for the quantum (blue squares) and classical (red circles) GoL as a function of the initial population density ρ_0. Simulations are performed with a t-DMRG at third order, Trotter step $\delta t = 10^{-2}$, truncation dimension $m = 30$, size $L = 32$, averaged over up to thirty different initial configurations.

and P' of the positions of the n and $\bar{n}$ operators) and $\mathcal{N}^2$ ($\mathcal{N}^3$) gives the null operator if the population is different from two (three), the identity otherwise. For classical states, as for example an initial random configuration of dead and alive states, the Hamiltonian (1) is, at time zero, $H_{Active} = b_i + b_i^\dagger$ on the sites with two or three alive neighbours and $H_{Hibernate} = 0$ otherwise. If the Hamiltonian would remain constant, every active site would oscillate forever while the hibernated ones would stand still. On the contrary as soon as the evolution starts, the state evolves into a superposition of possible classical configurations, resulting in a complex dynamics as shown below and the interaction between sites starts to play a role. Thus, the Hamiltonian introduced in Eq. (1) induces a quantum dynamics that resembles the rules of the GoL: a site with less than two or more than three alive neighbouring sites "freezes" while, on the contrary, it "lives". The difference with the classical game – connected to the reversibility of quantum dynamics – is that "living" means oscillating with a typical timescale between two possible classical states (see e.g. Fig. 1.

Dynamics. – To study the quantum GoL dynamics we employ the time dependent Density Matrix Renormalization group (DMRG). Originally developed to investigate condensed matter systems, the DMRG and its time dependent extension have been proven to be a very powerful method to numerically investigate many-body quantum systems [9–12]. As it is possible to use it efficiently only in one-dimensional systems, we concentrate to the one-dimensional version of the Hamiltonian (1): the operators $\mathcal{N}^2$ and $\mathcal{N}^3$ count the populated sites on the nearest-neighbour and next-nearest-neighbour sites and thus $\alpha = i-2, \beta = i-1, \gamma = i+1, \delta = i+2$. Note that it has been shown that the main statistical properties of the classical GoL are the same in both two- and one-dimensional versions [6].

To describe the system dynamics we introduce different quantities that characterise in some detail the system evolution. We first concentrate on the population dynamics, measuring the expectation values of the number operator at every site $\langle n_i(t) \rangle$. This clearly gives a picture of the "alive" and "dead" sites as a function of time, as it gives the probability of finding a site in a given state when measured. That is, if we observe the system at some final time T_f we will find dead or alive sites according to these probabilities. In Fig. 2 we show three typical evolutions (leftmost pictures): configuration A corresponds to a "blinker" where two couples of nearest-neighbour sites oscillate regularly between dead and alive states (a schematic representation of the resulting dynamics of the discretised population $\mathcal{D}_i(t)$ is reproduced also in Fig. 3); configuration B is a typical overcrowded scenario where twenty-four "alive" sites disappear leaving behind only some residual activity; finally a typical initial random configuration (C) is shown. Notice that in all configurations it is possible to identify the behaviour of the wave function tails that propagate and generate interference effects. These effects can be highlighted by computing the visibility of the dynamics, the maximum variation of the populations within subsequent generations, defined as:

$$v_i(t) = |\max_{t'} n_i(t') - \min_{t'} n_i(t')|; t' \in [t - \frac{T}{2}; t + \frac{T}{2}]; \quad (2)$$

that is, the visibility at time t reports the maximum variation of the population in the time interval of length T centered around t. The visibility clearly follows the preceding dynamics (see Fig. 2, second column) and identifies the presence of "activity" in every site.

To stress the connections and comparisons with the original GoL we introduce a classical figure of merit (shown in the third column of Fig. 2): we report a discretized version of the populations as a function of time ($\mathcal{D}_i(t) = 1$ for $n_i(t) > 0.5$ and $\mathcal{D}_i(t) = 0$ otherwise). Notice that $\mathcal{D}_i(t)$ gives the most probable configuration of the system

after a measurement on every site in the basis $\{|0\rangle, |1\rangle\}$. Thus, we recover a "classical" view of the quantum GoL with the usual definition of site status. For example, configuration A is a "blinker" that changes status at every generation (see Fig.2 and 3). More complex configurations appear in the other two cases. The introduction of the discretized populations $\mathcal{D}_i$ can also be viewed as a new definition of "alive" and "dead" sites from which we could have started from the very beginning to introduce a stochastic component as done in [8]. This quantity allows analysis to be performed as usually done on the classical GoL and to stress the similarities between the quantum and the classical GoL. Following the literature to quantify such complexity, we compute the clustering function $\mathcal{C}(\ell, t)$ that gives the number of clusters of neighbouring "alive" sites of size ℓ as a function of time [6]. For example, the function $\mathcal{C}(\ell, t)$ for a uniform distribution of "alive" sites would be simply $\mathcal{C}(L) = 1$ and zero otherwise while a random pattern would result in a random cluster function. This function characterises the complexity of the evolving patterns, e.g. it is oscillating between zero- and two-size clusters for the initial condition A, while it is much more complex for the random configuration C (see Fig. 2, rightmost column).

Statistics. – To characterise the statistical properties of the quantum GoL we study the time evolution of different initial random configurations as a function of the initial density of alive sites. We concentrate on two macroscopic quantities: the density of the sites that if measured would with higher probability result in "alive" states

$$\rho(t) = \sum_i \mathcal{D}_i(t)/L; \tag{3}$$

and the diversity

$$\Delta(t) = \sum_\ell \mathcal{C}(\ell, t), \tag{4}$$

the number of different cluster sizes that are present in the systems, that quantifies the complexity of the generated dynamics [6, 8]. Typical results, averaged over different initial configurations, are shown in Fig. 4 (left). As it can be clearly seen the system equilibrates and the density of states as well as the diversity reach a steady value. This resembles the typical behaviour of the classical GoL where any typical initial random configuration eventually equilibrates to a stable configuration. Moreover, we compare the quantum GoL with a classical reversible version of GoL corresponding to that introduced here: at every step a cell changes its status if and only if within the first four neighbouring cells only three or two are alive. Notice that, the evolution being unitary and thus reversible, the equilibrium state locally changes with time, however the macroscopic quantities reach their equilibrium values that depend non trivially only on the initial population density. In fact, for the classical game, we were able to check that the final population density is independent of the system size while the final diversity scales as $L^{1/2}$ (up to 2^{10} sites, data not shown). Moreover, the time needed to reach equilibrium is almost independent of the system size and initial population density. These results on the scaling of classical system properties support the conjecture that our findings for the quantum case will hold in general, while performing the analysis for bigger system sizes is highly demanding. A detailed analysis of the size scaling of the system properties will be presented elsewhere. In Fig. 4 we report the final (equilibrium) population density (right upper) and diversity (right lower) as a function of the initial population density for both the classical and the quantum GoL for systems of $L = 32$ cells. The equilibrium population density ρ is a non linear function of the initial one ρ_0 in both cases: the classical one has an initial linear dependence up to half-filling where a plateau is present up to the final convergence to unit filling for $\rho_0 = 1$. Indeed, the all-populated configuration is a stable system configuration. The quantum GoL follows a similar behaviour, with a more complex pattern. Notice that here a first signature of quantum behaviour is present: the steady population density reached by the quantum GoL is always smaller than its classical counterpart. This is probably due to the fact that the evolution is not completely captured by this classical quantity: the sites with population below half filling, i.e. the tails of the wave functions, are described as unpopulated by $\mathcal{D}_i$. However, this missing population plays a role in the evolution: within the overall superposition of basis states, a part of the probability density (corresponding to the states where the sites are populated) undergoes a different evolution than the classical one. In general, the quantum system is effectively more populated than the classical ρ indicates. This difference in the quantum and classical dynamics is even more evident in the dependence of the equilibrium diversity on the initial population density ρ_0. In the classical case the maximum diversity is slightly above three: on average, in the steady state, there are no more than about three different cluster sizes present in the system independently of the initial configuration. On the contrary –in the quantum case– the maximal diversity is about four, increasing the information content (the complexity) generated by the evolution by about $10 - 20\%$.

These findings are a signature of the difference between quantum and classical GoL. In particular we have shown that the quantum GoL has a higher capacity of generating diversity than the corresponding classical one. This property arises from the possibility of having quantum superpositions of states of single sites. Whether purely quantum correlations (entanglement) play a crucial role is under investigation. Similarly, as there is some arbitrariness in our definition of the quantum GoL, the investigation of possible variations is left for future work.

The investigation presented here fits perfectly as a subject of study for quantum simulators, like for example cold atoms in optical lattices. Indeed, the five-body Hamiltonian (1) can be written in pseudo spin-one-half operators

(Pauli matrices) and thus it can be simulated along the lines presented in [13]. In particular, these simulations would give access to investigations in two and three dimensions that are not feasible by means of t-DMRG [10].

In conclusion we note that this is one of the few available simulations of a many-body quantum game scalable in the number of sites [14–16]. With a straightforward generalisation (adding more than one possible strategy defined in Eq. (1)) one could study also different many-player quantum games. This approach will allow different issues to be studied related to many-player quantum games such as the appearance of new equilibria and their thermodynamical properties. Moreover, the approach introduced here shows that one might investigate many different aspects of many-body quantum systems with the tools developed in the field of complexity and dynamical systems: In particular, the relations with Hamiltonian quantum cellular automata in one dimension and quantum games [14,17]. Finally, the search for the possible existence of self-organised criticality in these systems along the lines of similar investigations in the classical GoL [18], if successful, would be the first manifestation of such effect in a quantum system and might have intriguing implications in quantum gravity [19,20].

After completing this work we became aware of another work on the same subject [21].

We acknowledge interesting discussions and support by R. Fazio and M.B. Plenio, the SFB-TRR21, the EU-funded projects AQUTE, PICC for funding, the BW-Grid for computational resources, and the PwP project for the t-DMRG code (www.dmrg.it).

REFERENCES

[1] M. Gardner, Sci. Am. **223**, 120 (1970).
[2] "Winning ways for your mathematical plays" J. Conway *et.al.*, A K Peters/CRC Press (1982).
[3] "Game of Life cellular Automata", Andrew Adamatzky (ed.), Springer 2010.
[4] "Collision-Based Computing", A. Adamatzky (ed.), Springer (2002).
[5] M. Dresden and D. Wong, Proc. Nat. Acad. Sci. USA **72**, 956 (1975).
[6] T.R.M. Sales, J.Phys. A **26**, 6187 (1993).
[7] "Quantum aspects of life", A.P. Flitney and D. Abbot (ed.), Imperial College Press (2008).
[8] L.S. Schulman and P.E. Seiden, J. Stat. Phys. **19**, 293 (1978).
[9] S.R. White and A.E. Feiguin, Phys. Rev. Lett. **93**, 076401 (2004).
[10] U. Schollwöck Rev. Mod. Phys. **77**, 259 (2005); K. Hallberg Adv. Phys. **55**, 477 (2006).
[11] A.J. Daley, C. Kollath, U. Schollwöck and G. Vidal, J. Stat. Mech.: Theor. Exp. P04005 (2004).
[12] G. De Chiara, M. Rizzi, D. Rossini, and S. Montangero, J. Comput. Theor. Nanosc. **5**, 1277 (2008).
[13] M. Lewenstein *et. al.*, Adv. Phys. **56**, 243 (2007); E. Jan *et. al.*, Quantum Inf. Comput. **3**, 15 (2003); J.J. Garcia-Ripoll, A.M. Martin-Delgado, and J.I. Cirac, Phys. Rev. Lett. **93**, 250405 (2004); L.M. Duan, E. Demler, and M.D. Lukin, Phys. Rev. Lett. **91**, 090402 (2003).
[14] J. Eisert, M. Wilkens, and M. Lewenstein, Phys. Rev. Lett. **83**, 3077 (1999).
[15] S.C. Benjamin and P.M. Hayden, Phys. Rev. A **64**, 030301 (2001).
[16] Q. Chen, Y.Wang, J-T Liu, and K-L Wang, Phys. Lett. A **327**, 98 (2004).
[17] D. Nagaj and P. Wocjan, Phys. Rev. A **78**, 032311 (2008).
[18] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987). P. Bak, K. Chen, and M. Creutz, Nature **342**, 780 (1989). C. Bennet and M.S. Bowutschky, Nature **350**, 468 (1991).
[19] M.H. Ansari and L. Smolin, Class. Quantum Grav. **25**, 095016 (2008).
[20] R. Borissov and S. Gupta, Phys. Rev. D **60**, 024002 (1999).
[21] P. Arrighi and J. Grattage, Proceedings of JAC 2010 - Journes Automates Cellulaires 2010, Finland (2010).